

ERGODICITY OF ITERATED FUNCTION SYSTEMS VIA MINIMALITY ON THE HYPER SPACES

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ABSTRACT. We give a sufficient condition for the ergodicity of the Lebesgue measure for an iterated function system of diffeomorphisms. This is done via the induced iterated function system on the space of continuum (which is called hyper-space). We introduce a notion of minimality for induced IFSs which implies that the Lebesgue measure is ergodic for the original IFS. Here, to beginning, the required regularity is C^1 . However, it is proven that the C^1 -regularity is a redundant condition to prove ergodicity with respect to the class of quasi-invariant measures.

As a consequence of mentioned results, we obtain ergodicity with respect to Lebesgue measure for several systems.

1. INTRODUCTION

The word “*Ergodic*” comes from classical statistical mechanics. In the context of dynamical systems, ergodic theory is the statistical study of systems relative to at least quasi-measures. Actually, ergodicity is concerned with the behavior of null and conull invariant sets.

Nowadays, the existence of a relationship between minimality and ergodicity can not be denied. This relation between minimality and ergodicity yields to the following question: Under what conditions a minimal iterated function system on a compact differentiable manifold is volume-ergodic? For instance, by $C^{1+\alpha}$ -regularity and expanding assumptions, Navas proved that minimality implies Lebesgue ergodicity for a group action on the circle (Ref. [13]). But the generalization result of Navas to higher-dimensional manifolds seems to have a much more complicated structure. In Ref. [3], by additional assumption of conformality of generators, authors obtained Lebesgue ergodicity for a group action on higher-dimensional manifolds. We refer to Ref. [15], for a robust example of conformal minimal systems, in dimension two.

On the other hands, in Ref. [14], it is known that the C^1 -regularity is not a sufficient condition to conclude ergodicity from minimality. More interesting cases are taken into consideration concern to the context of $C^{1+\alpha}$ -diffeomorphisms (C^1 -diffeomorphisms with α -Hölder derivatives with $\alpha > 0$).

Key words and phrases. iterated function systems, induced iterated function systems, ergodicity, minimality, hyper-space, induced map.

Summing up what previously mentioned: it seems that $C^{1+\alpha}$ -regularity and some kinds of hyperbolicity (e.g. uniform, non-uniform or partial) play essential role to prove ergodicity.

The aim of present work is to establish the ergodicity for some C^1 -regular systems which can be far from hyperbolicity, in some sense. On the other hand, it seems that ergodicity was never studied via the induced iterated function systems on hyper-spaces. It implies that we study the problem from a different point of view.

1.1. Minimality and Ergodicity of iterated function systems. An iterated function system generated by finitely many maps is the collection of all possible compositions of the maps which is a popular way to generate and explore a variety of fractals. More precisely, consider finitely many maps $\mathcal{F} = \{f_1, \dots, f_k\}$ on a compact metric space (X, d) . Write $\langle \mathcal{F} \rangle^+$ for the semigroup generated by collection \mathcal{F} under function composition. The action of the semigroup $\langle \mathcal{F} \rangle^+$ on X is called the iterated function system associated to \mathcal{F} and we denote it by $\text{IFS}(X; \mathcal{F})$ or $\text{IFS}(\mathcal{F})$.

We said that the $\text{IFS}(X; \mathcal{F})$ is minimal if every invariant non-empty closed subset of X is the whole space X , where $A \subset X$ is invariant for $\text{IFS}(X; \mathcal{F})$ if

$$f(A) \subseteq A; \quad \forall f \in \mathcal{F}.$$

In additional, let X equipped to a quasi-invariant probability measure μ . The $\text{IFS}(X; \mathcal{F})$ is *ergodic* with respect to μ if $\mu(A) \in \{0, 1\}$ for all invariant set A of $\text{IFS}(X; \mathcal{F})$.

Observe that the counterpart to ergodicity is minimality, in topological point of view. So, it is logical that the relation between Ergodicity and minimality have been studied extensively by many authors; (see for instance see Refs. [3], [4], [7], [14] and [15]).

As previously mentioned, we will focus on the Lebesgue measure which is quasi-invariant for C^1 (local) diffeomorphisms.

1.2. Induced maps on hyper-spaces. Let $f : X \rightarrow X$ be a continuous map from a continuum (that is, a compact connected metric space) X into itself. Consider the hyper-space

$$\mathcal{K}(X) = \{A \subseteq X : A \text{ is non - empty compact connected subset of } X\}$$

with the Hausdorff metric d_H on it which denoted by $(\mathcal{K}(X), d_H)$. It is known that if X is compact, so is the hyperspace $\mathcal{K}(X)$ (see Ref. [12]). A induced map is defined as follow

$$\hat{f} : \mathcal{K}(X) \rightarrow \mathcal{K}(X), \quad \hat{f}(A) \stackrel{\text{def}}{=} f(A) = \bigcup_{a \in A} \{f(a)\}.$$

Consider a classical dynamical system (J, f) where f is continuous and J is one-dimensional continuum. Although, the systems (J, f) and $(\mathcal{K}(J), \hat{f})$ have a similar behavior in some phenomena (see Refs. [5], [6] and [11]), but there exist some other phenomena in which the behavior of these systems are completely different. More precisely, the induced map is never transitive, even if f happens to be Refs. [1] and [9]. Clearly, the induced map is never minimal, too.

The main goal of this note is study of iterated function systems generated by finite induced maps to getting ergodicity of original system.

1.3. Induced IFSs and ergodicity of original IFSs. Now, consider $\text{IFS}(X; \mathcal{F})$ with $\mathcal{F} = \{f_1, \dots, f_k\}$. Naturally, we can define the induced iterated function system $\text{IFS}(\mathcal{K}(X); \hat{\mathcal{F}})$ on the space $(\mathcal{K}(X), d_H)$ in which $\hat{\mathcal{F}} = \{\hat{f}_1, \dots, \hat{f}_k\}$ and \hat{f}_i is induced map, for $1 \leq i \leq k$.

As previously mentioned, induced map is never minimal or even transitive. So, logically, the following question arises.

Question 1. Is there $\text{IFS}(X; \mathcal{F})$ so that $\text{IFS}(\mathcal{K}(X); \hat{\mathcal{F}})$ is minimal or transitive?

In this note, independent of the answer of above question, we define a weak kind of minimality for induced iterated function systems on hyper-space.

Definition 1.1. For $\Theta > 5$, we say that $\text{IFS}(X; \mathcal{F})$ is a Θ -hyper-minimal if for some $r_0 > 0$ and every $x, y \in X$ the following hold: for every $0 < r < r_0$ there exists $\hat{h} = \hat{h}(r, x, y) \in \text{IFS}(\mathcal{K}(X); \hat{\mathcal{F}})$ so that

$$d_H(\hat{h}(B(x, r)'), B(y, r')) < r/\Theta,$$

where $B(x, r)$ is the geodesic ball of radius r centered about x and $B(x, r)'$ is the set of all limit points of the set $B(x, r)$.

In many works to prove ergodicity, bounded distortion is a main ingredient in the proof. An important things about Θ -hyper-minimal is that bounded distortion is inherent in the definition of Θ -hyper-minimal. Then there is no reason to restrict our generators to especial maps (contracting or expanding maps), when we do not have to apply the classical bounded distortion (compare with Refs. [3] and [15]).

Now, we are ready to formulate the main result of this paper.

Theorem A. Suppose that M is a smooth compact differential manifold and the $\text{IFS}(M; \mathcal{F})$ is Θ -hyper-minimal, for $\mathcal{F} = \{f_1, \dots, f_k\} \subset \text{Diff}^1(M)$. Then the system $\text{IFS}(M; \mathcal{F})$ is ergodic with respect to Lebesgue measure.

The next result states a generalized form of Theorem A.

Corollary B. Every Θ -hyper-minimal IFS generated by finitely many homeomorphisms, which the Lebesgue measure is quasi-invariant¹ for them, is ergodic with respect to Lebesgue measure.

As another consequence of Theorem A, we obtain ergodicity for some generic ordinary systems. Let

$$\text{CCT}_m(\mathbb{T}^2) = \{hR_\alpha h^{-1} : h \in \text{Home}_m(\mathbb{T}^2), \alpha \in \mathbb{T}^2\},$$

where $\text{Home}_m(\mathbb{T}^2)$ consists of all homeomorphisms that the Lebesgue measure m is quasi-invariant with respect to them and R_α for $\alpha = (\alpha_1, \alpha_2)$ is the rigid translation

$$R_\alpha = R_{(\alpha_1, \alpha_2)} : \mathbb{T}^2 \rightarrow \mathbb{T}^2; (x, y) \mapsto (x + \alpha_1, y + \alpha_2).$$

¹A measure μ is said to be quasi-invariant if $(f_i)_* \mu$ is absolutely continuous with respect to μ for every $i = 1, \dots, k$.

We denote the closure of the set $CCT_m(\mathbb{T}^2)$ in the C^0 -topology by $\overline{CCT_m}^0(\mathbb{T}^2)$.

Corollary C. *Ergodicity with respect to Lebesgue measure in the space $\overline{CCT_m}^0(\mathbb{T}^2)$ is a generic property.*

We remark that the metric entropy of every elements of $\overline{CCT_m}^0(\mathbb{T}^2)$ is zero.

2. PROOF OF MAIN RESULT: THEOREM A

In this section, we prove Theorem A and also, we state and prove another form of it. To this end, we begin with the following definition.

Definition 2.1. *Consider two different positive numbers $0 \leq t \leq 1/2$, $0 < \ell < 1$ and also, consider $\Theta > 1$. We say that Θ is a (t, ℓ) -overlap number on a smooth compact differential manifold M if for every $r > 0$ and $x \in M$ the following hold:*

$$m(B(x, r) \cap B(y, r - r/\Theta)) - tm(B(x, r)) > \ell m(B(x, r - r/\Theta)); \quad \forall y \in B(x, r/\Theta), \quad (1)$$

where m is normalized Lebesgue measure

Observe that if $1/\Theta$ are necessary close to zero, one can ensure that always exist t, ℓ in which $t, 1 - \ell$ are sufficiently closed to zero and Θ is a (t, ℓ) -overlap number. In fact, in Definition 1.1, we put $\Theta > 5$ for existence suitable numbers t, ℓ so that Θ is a (t, ℓ) -overlap number. Thus, hereafter, we take $\Theta > 5$.

2.1. Global dynamics. Before we start to prove Theorem A, notice that if the $\text{IFS}(M; \mathcal{F})$ is Θ -hyper-minimal, then the system $\text{IFS}(M; \mathcal{F})$ is minimal. Indeed, let x, y be two arbitrary points in M and $B(y, a)$ be a neighborhood of y . By the assumption, one can find $r < a/2$ and $\hat{h} \in \text{IFS}(\mathcal{K}(M); \hat{\mathcal{F}})$ so that $d_H(\hat{h}(B(x, r)'), B(y, r')) < r/\Theta < r/4$, for some $\Theta > 5$. Thus $\hat{h}(B(x, r)) \subset B(y, a)$, which implies that minimality of $\text{IFS}(M; \mathcal{F})$.

Proof of Theorem A. Suppose that M is a smooth compact differential manifold and m is normalized Lebesgue measure. Also, assume that $\text{IFS}(M; \mathcal{F})$ is Θ -hyper-minimal. Since $\Theta > 5$, there are t, ℓ so that Θ is a (t, ℓ) -overlap number. Suppose that $0 < m(\mathcal{B}) < 1$ and $f_i(\mathcal{B}) \subset \mathcal{B}$ for all $i = 1, \dots, k$. It implies that $h(\mathcal{B}) \subset \mathcal{B}$, for every $h \in \text{IFS}(M; \mathcal{F})$. So, one can have $h(\mathcal{B} \cap B(p, r)) \subseteq \mathcal{B} \cap h(B(p, r))$.

One can choose a point $p \in DP(\mathcal{B})$ when $0 < m(\mathcal{B}) = m(DP(\mathcal{B}))$. Hence there is $\kappa_0 > 0$ so that for every $0 < \kappa < \kappa_0$

$$m(\mathcal{B} : B(p, \kappa)) = \frac{m(\mathcal{B} \cap B(p, \kappa))}{m(B(p, \kappa))} \geq 1 - t.$$

Since $\text{IFS}(M; \mathcal{F})$ is Θ -hyper-minimal, by definition, for some $0 < r_0 < k_0$ and every $x \in M$ the following hold: for every $0 < r < r_0$ there exists $\hat{h} = \hat{h}(r, p, x) \in \text{IFS}(\mathcal{K}(M); \hat{\mathcal{F}})$ so that

$$d_H(\hat{h}(B(p, r)'), B(x, r')) < r/\Theta,$$

and it implies that $\hat{h}(B(p, r)) \subseteq B(x, r + r/\Theta)$. In particular, it holds for $x \in B(p, r/\Theta)$. On the other hand, Θ is a (t, ℓ) -overlap number. Thus,

$$m(B(p, r) \cap B(x, r - r/\Theta)) - tm(B(p, r)) > \ell m(B(p, r - r/\Theta)),$$

for $x \in B(p, r/\Theta)$. It follows that

$$\ell m(B(p, r - r/\Theta)) < m(\hat{h}(B(p, r)) \cap \mathcal{B}) < m(B(x, r + r/\Theta)).$$

Take $J = B(x, r)$. Hence, one can have

$$\begin{aligned} \frac{m(\mathcal{B} \cap J)}{m(J)} &\geq \frac{\frac{\Theta}{\Theta+1} m(\mathcal{B} \cap \hat{h}(B(p, r)))}{m(J)} \\ &\geq \frac{\Theta}{\Theta+1} \cdot \frac{m(\mathcal{B} \cap \hat{h}(B(p, r)))}{m(J)} \\ &> \frac{\Theta}{\Theta+1} \cdot \frac{\ell m(B(p, r - r/\Theta))}{m(J)} \\ &> \frac{\Theta}{\Theta+1} \cdot \ell \cdot \frac{m(B(p, r - r/\Theta))}{m(J)} \\ &> \frac{\Theta}{\Theta+1} \cdot \ell \cdot c \end{aligned}$$

where the constant c is equal to $\frac{m(B(p, r - r/\Theta))}{m(J)}$. This means that $\frac{m(\mathcal{B} \cap J)}{m(J)}$ is bounded from below for every neighborhood J of x . It is equivalent to: for every $x \in M$, $x \notin DP(\mathcal{B}^c)$ which is a contradiction, when $\text{IFS}(M; \mathcal{F})$ is minimal and the Lebesgue measure is a quasi-invariant for any of generators. \square

Remark 2.2. In the proof of Theorem A, we used the C^1 -regularity just to show that the Lebesgue measure is quasi-invariant, so the proof of Corollary B is similar to Theorem A.

2.2. Local dynamics. Its visible from the proof of Theorem A, the definition 1.1 has more than which we need to proof ergodicity. So, one can define a locally version of its.

Definition 2.3. We say that $\text{IFS}(M; \mathcal{F})$ is locally Θ -hyper-minimal if there exists an open set $U \subseteq M$, measurable subset U' of U with $m(U') > 0$ and $r_0 > 0$ so that for all $x \in U'$ and $y \in U$ in which the following holds: for every every $0 < r < r_0$, there exists $\hat{h} = \hat{h}(r) \in \text{IFS}(\mathcal{K}(M); \hat{\mathcal{F}})$ so that

$$d_H(\hat{h}(B(x, r)'), B(y, r')) < r/\Theta.$$

Corollary 2.4. Suppose that M is a smooth compact differential manifold and $\text{IFS}(M; \mathcal{F})$ is minimal and locally Θ -hyper-minimal, for $\mathcal{F} = \{f_1, \dots, f_k\} \subset \text{Diff}^1(M)$. Then the system $\text{IFS}(M; \mathcal{F})$ is ergodic with respect to Lebesgue measure.

Proof. Suppose that $0 < m(\mathcal{B}) < 1$ and $f_i(\mathcal{B}) \subset \mathcal{B}$ for all $i = 1, \dots, k$. Also, suppose that $U \subseteq M$ is an open set and U' is a measurable subset U with $m(U') > 0$ so that they satisfy in Definition 2.3. On the other hand, $m(DP(\mathcal{B}^c)) + m(DP(\mathcal{B})) = 1$ and $0 < m(\mathcal{B})m(\mathcal{B}^c) < 1$.

Thus, $U' \cap DP(B^c) \neq \emptyset$ or $U' \cap DP(B) \neq \emptyset$. Without loss of generality, let $p \in U' \cap DP(B) \neq \emptyset$. Hence there is $\kappa_0 > 0$ so that for every $0 < \kappa < \kappa_0$

$$m(\mathcal{B} : B(p, \kappa)) = \frac{m(\mathcal{B} \cap B(p, \kappa))}{m(B(p, \kappa))} \geq 1 - t.$$

Let y arbitrary point U , by assumption, for every $r < r_0$ with $r_0 < k_0$ there exists $\hat{h} \in \text{IFS}(\mathcal{K}(M); \hat{\mathcal{F}})$ so that

$$d_H(\hat{h}(B(p, r)'), B(y, r')) < r/\Theta.$$

By similar argument used in the proof of Theorem A, for $r < r_0$ one can prove that $\frac{m(\mathcal{B} \cap B(y, r))}{m(B(y, r))}$ is bounded from below for every $y \in U$. It is equivalent to: for every $y \in U$, $y \notin DP(\mathcal{B}^c)$. Then $m(U' \cap DP(B)) = m(U)$ which is a contradiction, when $\text{IFS}(M; \mathcal{F})$ is minimal and the Lebesgue measure is a quasi-invariant for any of generators. \square

3. EXAMPLES

We are going to consider several examples, which all of them are ergodic with respect to Lebesgue measure.

3.1. Example on the circle. In this example, we construct an iterated function systems on hyper-space over the circle. To this end, let I_1, I_2 be two open connected subsets of S^1 with the following property

- (i) $I_1 \cup I_2 = S^1$,
- (ii) $m(S^1 \setminus I_1) < \frac{1}{20}$,
- (iii) for all $x \in S^1 \setminus I_1$, $B(x, 1/4) \subset I_2$,

Let f_1, f_2 be two $C^{1+\alpha}$ function on the circle so that

$$f_1|_{I_1} = R_\beta, \quad f_2|_{I_2} = R_\gamma$$

where $\beta \approx \gamma \in \mathbb{Q}^c$, and $\beta \gtrsim m(S^1 \setminus I_1)$. Take $\mathcal{F} = \{f_1, f_2\}$. We claim that $\text{IFS}(\mathcal{K}(S^1); \hat{\mathcal{F}})$ is Θ -hyper-minimal. Since we work on one dimension, it means that for every $x, y \in S^1$ and some $r_0 > 0$ the following holds: for every $r < r_0$ there exists $\hat{h} \in \langle \hat{\mathcal{F}} \rangle^+$ in which

$$d_H(\hat{h}(B(x, r)), B(y, r)) < r/5.$$

Since $x \in I_1$ (resp. $x \in I_2$), one can apply f_1 (resp. f_2) and so, from this point of view, $f_\omega(x) = R_\omega(x)$ for some $\omega \in \Sigma_k^+$.

On the other hand, from [2], if ω is a dense sequence under the shift map then the fiberwise orbit dive by it is dense on the circle for all $x \in S^1$. However, from [8], for every $x \in S^1$, there exists $\omega \in \Sigma_k^+$ such that $O_\omega^+(x)$ is nowhere dense on the circle.

Take

$$k = \max\{n \in \mathbb{N}; n\beta \leq 1 - \gamma\} \in \mathbb{Q}.$$

Now, for x in X , without loss of generality, let $x \in I_1$. We apply f_1 on x . Again, if $f_1(x) \in I_1$ and $S^1 \setminus I_1 \not\subseteq [x, f_1(x)]$ apply f_1 on $f_1(x)$. Otherwise, if $f_1(x) \notin I_1$ or $S^1 \setminus I_1 \subseteq [x, f_1(x)]$ apply f_2 on $f_1(x)$. By an inductive process, one can construct $\omega \in \Sigma_k^+$ so that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Card}\{i \in \mathbb{N} : \omega_i = 1 \text{ and } i \leq n\} = \frac{k-1}{k},$$

which implies that $\overline{\mathcal{O}_\omega^+(x)} = X$. On the other hand, by construction ω , for every $i \in \mathbb{N}$, $f_\omega^i(x) = R_\omega^i(x)$. Therefore, there is $i \in \mathbb{N}$ so that $f_\omega^i(x)$ is sufficiently close to y . Also, when $f_\omega^i(x) = R_\omega^i(x)$ by choosing suitable i , one can have $d_H(B(y, r), f_\omega^i(B(x, r))) < r/5$ for every r less than the Lebesgue number of the open covering $\{I_1, I_2\}$.

Hence IFS($S^1; f_1, f_2$) is ergodic with respect to Lebesgue measure.

3.2. Example on the torus. Now, we prove that the Lebesgue measure is ergodic for some conjugacy class of translations of \mathbb{T}^2 . For $x \in M$, let a subset Γ_x of $\text{Home}(M)$ be defined as follow: the set of all $g \in \text{Home}(M)$ so that there exist an invertible matrix $A_{2 \times 2}$ and a neighborhood U_x of x with

$$g(z) = A(z - x) + g(x); \quad \forall z \in U_x$$

Now, let $\gamma \in \mathbb{Q}^c \times \mathbb{Q}^c \cap \mathbb{T}^2$ and $h \in \Gamma_{x_0}$, for some $x_0 \in \mathbb{T}^2$. Since R_γ is minimal, the conjugate of it $hR_\gamma h^{-1}$ is minimal, too. Thus, it is sufficient to prove that IFS($\mathcal{K}(\mathbb{T}^2); \widehat{hR_\gamma h^{-1}}$) is locally Θ -hyper-minimal, for a Θ greater than 5.

Since $h \in \Gamma_{x_0}$, one can find $0 < \rho_h$ so that $Dh(z) = Dh(y) = A$ for every $y, z \in B(x_0, \rho_h)$ and $Dh^{-1}(z) = Dh^{-1}(y) = A^{-1}$ for every $y, z \in h(B(x_0, \rho_h))$. Take $r_h = \rho_h/2$ and $y \in B(x_0, r_h)$. For $0 < r < r_h$ and using of construction of h, h^{-1} , it is not hard to find that there is n_j so that for $z, y \in B(x_0, r_h)$

$$d_H(hR_\gamma^{n_j} h^{-1}(B(z, r)'), B(y, r')) < r/\Theta.$$

Summing up, the following holds:

Corollary 3.1. *Let $\gamma \in \mathbb{Q}^c \times \mathbb{Q}^c \cap \mathbb{T}^2$ and $h \in \Gamma_{x_0}$, for some $x_0 \in \mathbb{T}^2$. Then IFS($\mathbb{T}^2; hR_\gamma h^{-1}$) is ergodic with respect to Lebesgue measure.*

3.3. Example on the sphere. Here, a sphere S^2 is defined as the set of points that are all at the same distance $r = \frac{1}{2\pi}$ from $0 = (0, 0, 0)$, in three-dimensional space.

Take a particular point $P = (0, 0, r)$ on a sphere as its north pole, then the corresponding antipodal point $Q = (0, 0, -r)$ is south pole. Notice that the equator S^1 of sphere, is great circle which lies in xy -space with length one. We equip S^1 to an orientation and we will denote by \widehat{ab} the arc from a to b according to this orientation. Also, the circle of sphere which through the point H and parallel to equator S^1 , denoted by $\Upsilon_{S^1}(H)$.

Observe that one can define for $\gamma \in (0, 1)$ a preserving orientation map $\Theta_\gamma : S^1 \rightarrow S^1$ as follows $\Theta_\gamma(a) = b$, where the length of the arc \widehat{ab} is equal to γ . It is well known that Θ_γ is minimal if γ is irrational.

Also, let $H = (x_1, y_1, z_1)$ be a point on 2-sphere. Take $\Xi(P, Q : H)$ the meridian longitude which contains points H, P, Q and also take $\Gamma(P, Q : H)$ the connected part of $\Xi(P, Q : H) \setminus \{P, Q\}$ which contains point H . Define a map $\pi_{\{P, Q\}} : S^2 \setminus \{P, Q\} \rightarrow S^1$ as follows $\pi_{\{P, Q\}}(H) = \Gamma(P, Q : H) \cap S^1$. Observe that the map $\pi_{\{P, Q\}}$ is not invertible but correspond to every point K in S^1 , one can correspond to K , $\psi_{\{P, Q\}}^H(K)$ on $\Upsilon_{S^1}(H)$ so that $\psi_{\{P, Q\}}^H(K) = \Upsilon_{S^1}(H) \cap \Gamma(P, Q : K)$ and

$$\psi_{\{P, Q\}}^H(\pi_{\{P, Q\}}(H)) = H.$$

The translations $T_\gamma : S^2 \rightarrow S^2$ have the form

$$T_\gamma(H) = \begin{cases} T_\gamma(H) = H, & H=p \text{ or } q; \\ T_\gamma(H) = \psi_{\{P, Q\}}^H(\Theta_\gamma(\pi_{\{P, Q\}}(H))), & \text{otherwise.} \end{cases}$$

Now, take a particular point $e = (r, 0, 0)$ on a sphere as its north pole, then the corresponding antipodal point $w = (-r, 0, 0)$ is south pole. For this additive notation, take \hat{S}^1 the equator of sphere which is great circle which lies in yz -space with length one. Again, we will equip \hat{S}^1 to an orientation and we will define $\hat{\Theta}_\gamma : \hat{S}^1 \rightarrow \hat{S}^1$ as follows $\hat{\Theta}_\gamma(x) = y$, where the length of the arc from x to y according to its orientation is γ .

The translations $R_\gamma : S^2 \rightarrow S^2$ have the form

$$R_\gamma(H) = \begin{cases} R_\gamma(H) = H, & H=e \text{ or } w; \\ R_\gamma(H) = \psi_{\{e, w\}}^H(\hat{\Theta}_\gamma(\pi_{\{e, w\}}(H))), & \text{otherwise.} \end{cases}$$

Now, consider the IFS generated by $\mathcal{F} = \{f_1, f_2\}$, where $f_1 = T_{\gamma_1}$ and $f_2 = R_{\gamma_2}$ for irrational numbers γ_1, γ_2 . It is not hard to show that the following hold

- i) Both of $\text{IFS}(S^2; \mathcal{F})$ and $\text{IFS}(S^2; \mathcal{F}^{-1})$ are minimal.
- ii) Both of $\text{IFS}(S^2; \mathcal{F})$ and $\text{IFS}(S^2; \mathcal{F}^{-1})$ is equicontinuous.
- iii) $\langle \mathcal{F} \cup \mathcal{F}^{-1} \rangle^+$ does not contain any minimal element.
- iv) The Lebesgue measure is ergodic for both of $\text{IFS}(S^2; \mathcal{F})$ and $\text{IFS}(S^2; \mathcal{F}^{-1})$.

Remark 3.2. Notice that both $\text{IFS}(S^2; \mathcal{F})$ and $\text{IFS}(S^2; \mathcal{F}^{-1})$ satisfies the deterministic chaos game by Theorem 3 in [10].

4. PROOF OF COROLLARY C: GENERIC ERGODICITY OF SOME SYSTEMS

Recall that a subset \mathcal{R} is residual if it contains a countable intersection of open dense sets. The space $C^0(\mathbb{T}^2)$ with the C^0 -topology is a Polish space, so that any closed subset of it (in particular, $\overline{CCT_m^0}(\mathbb{T}^2)$) is a Baire space. A property is said to be generic in a Baire space if it holds in a residual subset of the space.

Let us that we follow the notation of the Subsection 3.2. Observe that the set Γ_x is dense in the metric space $(\text{Home}(\mathbb{T}^2), d_0)$, for every $x \in \mathbb{T}^2$ and $\{R_\beta : \beta \in \mathbb{Q}^c \times \mathbb{Q}^c\}$ is dense in the set of all translations with the metric d_0 , then by Corollary 3.1 the following holds: the metric space $(\overline{CCT_m^0}(\mathbb{T}^2), d_0)$ contains of a dense subset which every element of it is ergodic with respect to the Lebesgue measure.

Moreover, for $h \in \Gamma_x$, we define $\mathcal{U}_n(h)$ consist of all $fR_\gamma f^{-1}$ for some $\gamma \in \mathbb{T}^2$ with the following properties

- i) $f \in \text{Home}(\mathbb{T}^2)$ and $\|f - h\|_1 < 1/n$,
- ii) $d_H(fR_\gamma^j f^{-1}(B(z, r)'), B(y, r')) < r/\Theta$; for $r_h < r < \frac{r_h}{n+1}$, $z, y \in B(x, r_h)$ and some $j \in \mathbb{N}$,

Since $\mathcal{U}_n(h)$ is a non-empty open set and Γ_x is dense in the space $(\text{Home}(\mathbb{T}^2), d_0)$, the set $\mathcal{U}_n = \cup_{h \in \Gamma_x} \mathcal{U}_n(h)$ is an open and dense subset of $(\overline{CCT_m^0}(\mathbb{T}^2), d_0)$. Take $\mathcal{U} = \cap_n \mathcal{U}_n$. It is clear that every $f \in \mathcal{U}$, is locally Θ -hyper-minimal, for a Θ greater than 5.

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